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Subcritical free-surface flow caused by a line source in a fluid of finite depth

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Abstract. An integral equation is derived and solved numerically to compute the flow and the free surface shape generated when water flows from a line source into a fluid of finite depth. At very low values of the Froude number, stagnation point solutions are found to exist over a continuous range in the parameter space. For each value of the source submergence depth to free stream depth ratio, an upper bound on the existence of stagnation point solutions is found. These results are compared with existing known solutions. A second integral equation formulation is discussed which investigates the hypothesis that these upper bounds correspond to the formation of waves on the free surface. No waves are found, however, and the results of the first method are confirmed.

1. Introduction

The physical, chemical and biological properties of water to be withdrawn from a reservoir, cooling pond or solar pond can only be predicted if the flow pattern induced by the withdrawal can be determined in advance. These water bodies are all examples in which the determination of these properties is of great importance for efficient or safe operation [13]. In all of these cases, the water is stratified in density, a factor which is crucial to determining the flow.

When the stratification is linear and the withdrawal rate is steady, the flow is quite well understood. A recent summary of this situation is given in Imberger and Patterson [14]. If the water body is stratified into layers of finite thickness, the flow is qualitatively understood, but many details remain unresolved. The actual situation is more complicated than either of these two cases since the stratification will have some more general form, and the flow is probably some combination of the two extremes. The work presented here examines the problem of withdrawal through a line sink from a single layer of finite depth with a free surface. Understanding this flow will shed more light on the case of withdrawal from distinct layers of fluid.

Experiments with two layers of fluid [11] suggest that for Froude numbers above some threshold value the free surface is drawn down directly into the sink. The flow conditions under which this occurs are still uncertain, despite the experimental and numerical work which has been carried out.

The flow can best be described in terms of a dimensionless parameter, the Froude number,

$$F_B = \left(\frac{q^2}{gh_B^3}\right)^{1/2} = \frac{U}{\sqrt{gh_B}}\,, \tag{1.1}$$

where q is the discharge from the source per unit width, h_B and U are the depth and velocity of the free stream respectively, and g is the acceleration due to gravity. If g is replaced by

$g' = (\Delta\rho/\rho)g$ in this and all subsequent equations, where ρ is the density of the layer, and $\Delta\rho$ is the difference in density between this layer and another above it, then the problem becomes a consideration of the shape of the interface between two layers when no fluid is being withdrawn from the upper layer.

In this paper, solutions are computed at the low Froude number end of the spectrum for flow from a line source into a single fluid of finite depth. The mathematical description of this problem is the same as that for a sink flow because of the quadratic dependence of the free surface condition upon the velocity, i.e. a sink flow has exactly the same solution but with the velocity reversed in direction along the streamlines.

Numerical evidence is presented which suggests that solutions with a stagnation point on the surface above the sink exist over all values of the submergence depth of the source, for Froude numbers up to some limiting value for each source depth. The results of a second formulation in which waves are possible on the free surface are discussed. No waves were found, however, giving qualitative agreement with the first method. These results are consistent with earlier results on problems with a slightly different geometry [5, 10].

When water flows from a line source into an homogeneous, inviscid, incompressible fluid with a free surface, Craya [1], Tuck and Vanden Broeck [18], Hocking [8] and Vanden Broeck and Keller [19] have shown, for several different geometries, that if the fluid is of infinite depth, there is a unique Froude number at which the free surface bends down into a cusp shape (see Fig. 1). In this case we must define the Froude number to be $F_S = (q^2/gh_S^3)^{1/2}$, where h_S is the depth of the source. At low values of the Froude number, Hocking and Forbes [10] have recently obtained numerical solutions in which a stagnation point forms on the free surface above the source (see Fig. 1). They were unable to obtain solutions of this type above a Froude number, (F_S) of about 1.4, suggesting a limit at this value. In a recent paper, Forbes and Hocking [7] found that when surface tension was included in the problem, solutions were limited by an upper bound in the Froude number

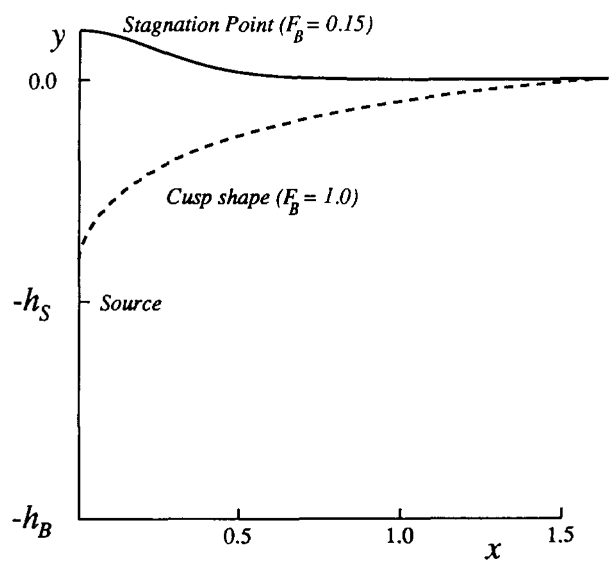

Fig. 1. Sketch of the problem under consideration. The cusped free surface was computed for the case $h_s/h_b = 0.5$ and $F_B = 1$ [12], and the stagnation point solution shown was computed for $h_S/h_B = 0.5$ and $F_B = 0.15$. The free surface elevation has been magnified by a factor of 10 for the stagnation point solution.

which occurred at a fold in the parameter space giving non-unique solutions at a single Froude number and source depth.

Previously, Peregrine [17], Vanden Broeck, Schwartz and Tuck [20], and Tuck and Vanden Broeck [18], had all attempted to compute solutions of this type, but were unable to compute solutions for F_S up to the limiting value. The corresponding cusped solution was found at $F_S = 3.56$ [18]. Tuck and Vanden Broeck [18] have shown that these are the only steady free surface configurations which are possible for this idealised problem.

Forbes and Hocking [5] have computed stagnation point solutions numerically in the case of axisymmetric flow into a point sink, for values of F_{3S} up to approximately 6.4. In three dimensions the Froude number is defined as $F_{3S} = (Q^2/gh_S^5)^{1/2}$ where Q is the total flux into the sink. At $F_{3S} = 6.4$, they found a secondary stagnation point formed on the free surface a small distance away from the primary point above the sink. As in the case of a line sink, no solutions were obtained for values of F_{3S} larger than this. No solutions of the cusped type have yet been found for the three dimensional, axisymmetric case.

If the layer is of finite depth, solutions with a cusp for flow from a line source have been found for all Froude numbers above a certain value, up to and including infinite Froude number [2, 9, 12, 15, 19]. In some cases this lower bound was found to be unity, in others it was found to be greater. In addition, a single branch of solution in which locally unique cusped flows which are dependent upon the height of the sink above the bottom were found for $F_B < 1.0$ (see Fig. 5). These results are discussed in Vanden Broeck and Keller [19], and Hocking [12]. Mekias and Vanden Broeck [16] have computed solutions with a stagnation point for values of F_B along a single branch in the parameter space which is bounded both above and below ($1 \leqslant F_B \leqslant 1.22$). Stagnation point solutions are not possible at infinite Froude number. Along this line in the parameter space both solution types appear to exist.

In section 2 of this paper, an integral equation will be derived which will allow computation of the flow under consideration. In section 3 the first term in a small Froude number expansion for the free surface angle will be calculated for comparison with the numerical scheme, which will be presented in section 4. An alternative formulation which was used in an attempt to compute waves on the free surface will be discussed in section 5. In the final section, these results will be placed in context with other work on this problem.

2. Problem formulation

The steady, irrotational motion of an inviscid, incompressible fluid in the presence of gravity is to be examined. The fluid is of infinite depth and has a free surface above a line source.

Let $z = x + iy$ be the physical plane, with the origin directly above the source and at the level of the free surface far away from the source, i.e. the level of the free stream (see Fig. 1). The mathematical problem is to find a complex potential $w(z) = \phi(x, y) + i\psi(x, y)$, which satisfies Laplace's equation ($\nabla^2 w = 0$) within the flow domain, conditions of no flow across the solid boundaries and the free surface, and the condition of constant pressure on the free surface, provided by Bernoulli's equation

$$p/\rho = gy + \frac{1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right) = \frac{1}{2}U^2, \tag{2.1}$$

on $y = \eta(x)$ where $\eta(x)$ is the equation of the interface shape, and U is the free stream velocity of the fluid far from the source. If we nondimensionalise with respect to the length $(m^2/8\pi^2 g)^{1/3}$ and the velocity $(mg/\pi)^{1/3}$, where m is the source strength, then this equation becomes

$$y + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left(\frac{\pi}{h_B}\right)^2 , \tag{2.2}$$

where h_B is the free stream depth of the fluid, the nondimensional flux from the source is π, and hence the velocity at large distances from the source is π/h_B. The flow is symmetric about $x = 0$, and consequently only the region $x \geq 0$ is considered. The Froude number based on the free stream velocity and depth, F_B is now given by $F_B = (2\pi^2/h_B^3)^{1/2}$. To be consistent with previous work the Froude number based on the source submergence depth, F_S, must be defined with double the flux into the region with $x > 0$ to account for the total flux into the whole symmetric region, and as a consequence takes the form $F_S = (8\pi^2/h_S^3)^{1/2}$, where h_S is the depth of the source beneath the free stream surface level.

To derive an integral equation for this problem we follow Hocking [12]. The transformation

$$e^w = \zeta \tag{2.3}$$

maps the infinite strip between $\psi = 0$ and $\psi = -\pi$ in the w-plane to the lower half of the ζ-plane. We choose to let $w = 0$ correspond to the stagnation point above the source, so that the free surface $\psi = 0$, $\phi > 0$, lies along the real ζ-axis where $\zeta \geq 1$. The source lies at the origin in the ζ-plane, and the negative real axis corresponds to $\psi = -\pi$ (see Fig. 2).

We seek w by solving for $\Omega(\zeta) = \delta(\zeta) + i\tau(\zeta)$, defined in relation to the complex conjugate of the velocity field by

$$w'(z(\zeta)) = \frac{\pi}{h_B} e^{-i\Omega(\zeta)} . \tag{2.4}$$

The magnitude of the velocity at any point is then given by $|w'(z)| = (\pi/h_B) e^{\tau(\zeta)}$, and the angle any streamline makes with the horizontal is $\delta(\zeta)$. For stagnation point solutions, we require that $\delta = 0$ at $\zeta = 1$ and $\delta \to 0$ as $\zeta \to \infty$.

The free surface corresponds to the positive real ζ-axis for $\zeta > 1$, and $\tau \to 0$ as $\zeta \to \infty$ to conserve volume. On the remainder of the real ζ-axis, which corresponds to the solid boundaries of the flow domain, the streamlines must be parallel to the walls, so that the condition that there be no flow normal to the solid boundaries is satisfied if $\delta(\zeta)$ is chosen to be the angle of the wall to the horizontal, i.e.

$$\delta(\zeta) = \begin{cases} 0 & \text{if } -\infty < \zeta < \zeta_B ; \\ -\pi/2 & \text{if } \zeta_B < \zeta < 0 ; \\ \pi/2 & \text{if } 0 < \zeta \leq 1 . \end{cases}$$

The only singularities of the function $\Omega(\zeta)$ in the ζ-plane are those at the origin, at $\zeta = 1$ and at $\zeta = \zeta_B$, corresponding to the source and the stagnation points above the source on the free surface and on the bottom beneath the source, respectively. All of these singularities can be shown to be weaker than a simple pole, so that Cauchy's Theorem can be applied to $\Omega(\zeta)$ on a path consisting of the real ζ-axis, a semi-circle at $|z| = \infty$ in the lower half plane, and a

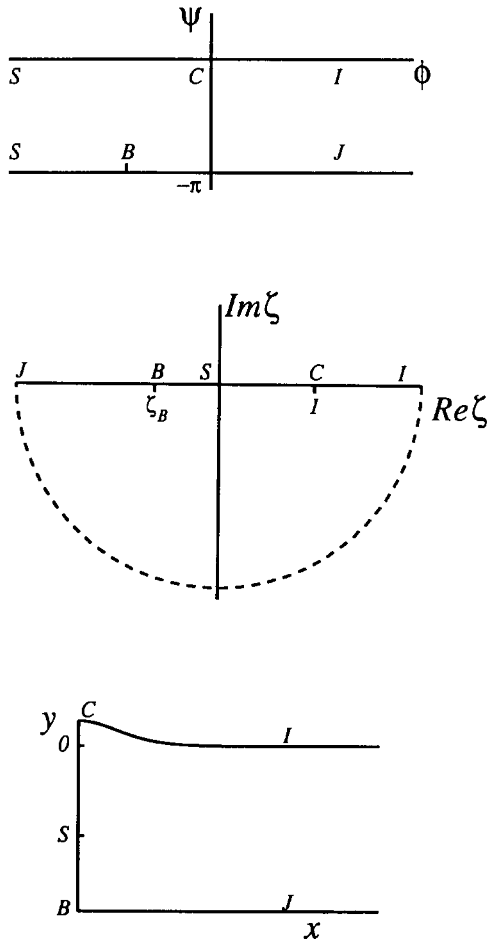

Fig. 2. Mapped planes used in the problem formulation; (a) the complex velocity potential w-plane, (b) the lower half ζ-plane, and (c) the physical z-plane.

circle of vanishing radius about the point ζ. Hence, for $\mathrm{Im}\{\zeta\} < 0$ we have

$$\Omega(\zeta) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\Omega(\zeta_0)}{\zeta_0 - \zeta}\, \mathrm{d}\zeta_0 , \tag{2.5}$$

since $\Omega \to 0$ as $|\zeta| \to \infty$. If we let $\mathrm{Im}\{\zeta\} \to 0^-$, we obtain

$$\tau(\zeta) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\delta(\zeta_0)}{\zeta_0 - \zeta}\, \mathrm{d}\zeta_0$$

and

$$\delta(\zeta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\tau(\zeta_0)}{\zeta_0 - \zeta}\, \mathrm{d}\zeta_0 , \tag{2.6}$$

where the integrals are of Cauchy–Principal Value form.

Substituting the known values of $\delta(\zeta)$ into the equation for $\exp[\tau(\zeta)]$ gives

$$\exp[\tau(\zeta)] = \left[\frac{(1-\zeta)(\zeta_B-\zeta)}{\zeta^2}\right]^{1/2} \exp\left[\frac{1}{\pi}\int_1^{+\infty}\frac{\delta(\zeta_0)}{\zeta_0-\zeta}\,\mathrm{d}\zeta_0\right]. \tag{2.7}$$

The equation for constant pressure on the free surface, which can be obtained by combining equations (2.2), (2.3) and (2.4) must be satisfied, giving

$$\frac{h_B}{\pi}\int_\infty^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\sin\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0 + \left(\frac{\pi}{h_B}\right)^2 \mathrm{e}^{2\tau(\zeta)} = \left(\frac{\pi}{h_B}\right)^2, \tag{2.8}$$

on $1 \leq \zeta < \infty$. This equation can be differentiated, rearranged and integrated to give the more convenient form

$$\exp[\tau(\zeta)] = \left[1 + \frac{3}{\pi F^2}\int_\zeta^\infty \frac{\sin\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0\right]^{1/3}, \tag{2.9}$$

on $1 \leq \zeta < \infty$. Combining (2.7) and (2.9) on the free surface gives a nonlinear integral equation for $\delta(\zeta)$ on $1 \leq \zeta < \infty$. The value of δ is known elsewhere on the boundary from the boundary conditions, and hence we can obtain τ from (2.6). Using δ and τ, it is possible to integrate (2.4) to obtain the location of points on the free surface. These may be written as

$$x(\zeta) = x(\zeta^*) + \frac{h_B}{\pi}\int_{\zeta^*}^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\cos\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0,$$

and

$$y(\zeta) = y(\zeta^*) + \frac{h_B}{\pi}\int_{\zeta^*}^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\sin\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0. \tag{2.10}$$

Since $y \to 0$ as $\zeta \to \infty$, the free surface condition requires that the height of the stagnation point is $(\pi/h_B)^2$, so that

$$\left(\frac{\pi}{h_B}\right)^2 = -\frac{h_B}{\pi}\int_1^\infty \frac{\mathrm{e}^{-\tau(\zeta_0)}\sin\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0. \tag{2.11}$$

The sink depth is

$$h_S = \frac{h_B}{\pi}\int_0^1 \frac{\mathrm{e}^{-\tau(\zeta_0)}}{\zeta_0}\,\mathrm{d}\zeta_0 - \left(\frac{\pi}{h_B}\right)^2, \tag{2.12}$$

and the base depth is

$$h_B = h_S + \frac{h_B}{\pi}\int_0^{\zeta_B} \frac{\mathrm{e}^{-\tau(\zeta_0)}}{\zeta_0}\,\mathrm{d}\zeta_0. \tag{2.13}$$

3. Asymptotic solution for small Froude number

At small values of the Froude number, we can obtain a solution to the equations generated by equating (2.7) and (2.9), by assuming that the angle of the free surface $\delta(\zeta)$ can be written as

$$\delta(\zeta_B;\zeta)=F_B^2\delta_2(\zeta_B;\zeta)+F_B^4\delta_4(\zeta_B;\zeta)+\cdots, \tag{3.1}$$

Substituting this expression into (2.7) and (2.9), and equating like powers of F_B, gives a set of equations,

$O(1)$:

$$1+\frac{3}{\pi}\int_\zeta^\infty \frac{\delta_2(\zeta_B;\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0=\left[\frac{(1-\zeta)(\zeta-\zeta_B)}{\zeta^2}\right]^{3/2}.$$

$O(F_B^2)$:

$$\int_\zeta^\infty \frac{\delta_4(\zeta_B;\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0=\left[\frac{(1-\zeta)(\zeta-\zeta_B)}{\zeta^2}\right]^{3/2}\int_1^\infty \frac{\delta_2(\zeta_B;\zeta_0)}{\zeta_0-\zeta}\,\mathrm{d}\zeta_0\,. \tag{3.2}$$

Expressions for the values of $\delta_i(\zeta_B;\zeta)$, $i=2,4,\ldots$ can be obtained by differentiating the successive equations. The first term is given by

$$\delta_2(\zeta_B;\zeta)=-\frac{\pi}{2}\,\frac{(1-\zeta)^{1/2}(\zeta-\zeta_B)^{1/2}}{\zeta^3}\,[\zeta_B(\zeta-2)+\zeta]\,. \tag{3.3}$$

Unfortunately, the next term has a logarithmic character, and the series can therefore only be treated as asymptotically valid. Nonetheless, it is still reasonable to compare it with our full numerical solutions for small values of the Froude number.

An interesting point is that the value of $\delta_2(\zeta_B;\zeta)$ is always negative if $\zeta_B>-1$, which indicates that the surface asymptotes to the free stream level from above. On the other hand, if $\zeta_B<-1$, $\delta_2(\zeta)$ is positive over that range of the interval which satisfies the condition $\zeta>|2\zeta_B/(1+\zeta_B)|$, which indicates that the free surface dips below the free stream level before asymptoting to it from beneath.

Figure 3 shows the expansion solution for the free surface angle (dashed lines) compared with the numerical solution for several cases in the range of interest. All of these examples are close to the limiting Froude number for these submergence depths, and the asymptotic result is beginning to diverge from the numerical solution. At smaller values of F_B, however, the two solutions are almost indistinguishable.

4. Numerical method

The nonlinear integral equation described by equations (2.7) and (2.9) can be solved by computing the integrals numerically at a set of discrete points and solving for $\delta(\zeta)$ using an iteration scheme.

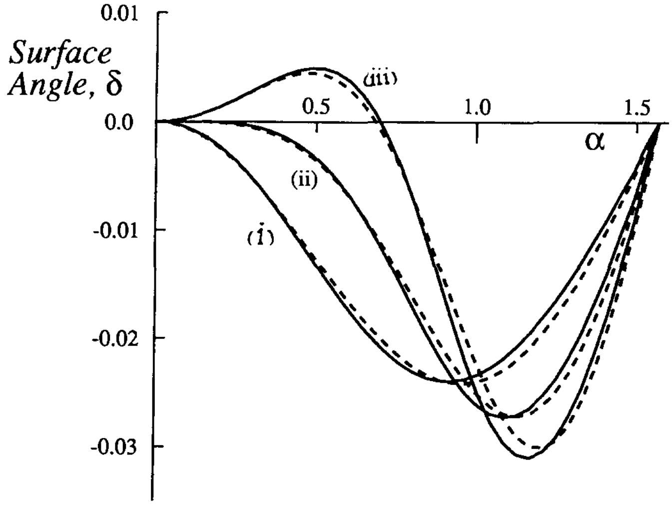

Fig. 3. The angle of the free surface plotted against the mapped variable α for (i) $F_B = 0.2$, $h_S/h_B = 1$, (ii) $F_B = 0.15$, $h_S/h_B = 0.5$ and (iii) $F_B = 0.075$, $h_S/h_B = 0.27$. The solid line is the nonlinear numerical solution, while the dashed line in each case is the linearised solution. All solutions are approaching the limiting value of F_B.

In order to avoid the difficulties which arise from the principal value integrals and the semi-infinite domain of integration, the following procedure was adopted. The range of ζ from 1 to infinity was mapped to $(0, \pi/2)$ using the transformation $\zeta = \sin^{-2}\alpha$. The integral equation then becomes

$$[1 - \zeta_B \sin^2\alpha]^{3/2} \exp\left[\frac{3}{\pi} \int_0^{\pi/2} \frac{\delta(\theta)\cos\theta}{(\sin^2\alpha - \sin^2\theta)\sin\theta}\, d\theta\right] = 1 + \frac{6}{\pi F_B^2} \int_\alpha^{\pi/2} \frac{\sin\delta(\theta)}{\tan\theta}\, d\theta\,. \quad (4.1)$$

The Cauchy–Principal Value integral was modified by letting $\delta(\alpha) = f(\alpha)\sin\alpha$ and adding and subtracting the value of f at $\alpha = \theta$ so that

$$\int_0^{\pi/2} \frac{\delta(\theta)\cos\theta}{(\sin^2\alpha - \sin^2\theta)\sin\theta}\, d\theta$$

becomes

$$-\int_0^{\pi/2} \frac{(f(\alpha) - f(\theta))\cos\theta}{(\sin^2\alpha - \sin^2\theta)}\, d\theta + \frac{f(\alpha)}{2\sin\alpha} \log\left[\frac{1+\sin\alpha}{1-\sin\alpha}\right],$$

which is no longer singular, and can be integrated numerically. All integration was performed using cubic splines on a variable mesh, so that points could be easily concentrated in the region of interest, in this case near the stagnation point. In all calculations the distribution of points was found to make little difference to the results, and an even distribution was used on most occasions. The interval was divided into N sub-intervals, $0 = \alpha_0, \alpha_1, \ldots, \alpha_N = \pi/2$. Equation (4.1) was evaluated at the internal mesh points, $\alpha_2, \ldots, \alpha_{N-1}$ giving $N-2$ equations for the $N-1$ unknowns $\delta_1, \ldots, \delta_{N-1}$. The equation at the omitted point was replaced by equation (2.11), which is equivalent to (4.1) evaluated at the stagnation point. The point α_1 corresponds to a point a long distance out on the free surface, and while this modification made no difference to the results, it significantly improved the convergence of the numerical scheme as N was increased because the correct condition was satisfied at the important stagnation point.

This system of nonlinear algebraic equations was solved for δ using a damped Newton iteration scheme, starting with a guess provided by the asymptotic series solution for small values of F_B, and halving the length of the correction vector if there was no decrease in the residual error at each step.

The numerical scheme converged rapidly for small values of F_B, usually taking only 3 iterations to satisfy the equations with an error of less than 10^{-10} at all points. As F_B was increased for each value of source depth a point was eventually reached at which the iterations failed to converge.

As F_B was increased this breakdown was signalled by the appearance of oscillations of wavelength equal to the size of the interval between the discrete points. These oscillations were clearly of numerical origin, and grew rapidly in amplitude as F_B was increased until the complete failure of the method. At each value of the submergence depth of the source the limiting value of F_B was taken as the largest value at which these numerically induced oscillations did not appear. The solutions computed were found to be accurate to 4 figures when $N = 240$, although there appeared to be a decrease in accuracy as $\zeta_B \to -\infty$.

5. The possibility of waves

It is well known that flows in a fluid of finite depth may have waves on the free surface, provided the free stream Froude number is less than one. It is therefore possible that the breakdown of the solution method presented in this paper is due to the formation of waves on the surface. The presence of such waves at $\alpha = 0$, which corresponds to the point at downstream infinity in the physical plane, would be a singularity of a particularly unpleasant type, and the method would surely fail.

In an attempt to overcome this difficulty, an alternative method was used which has in the past produced waves for some similar free surface flows [3, 4]. This method formulates the free surface flow problem as an integral equation in the physical plane using the arclength along the free surface as the independent variable. The details of this work are described in an internal report, Forbes and Hocking [6]. The important result of this work, however, is that despite the clear capability of this physical plane formulation to reproduce waves if they exist [3, 4], no waves were obtained.

The results obtained using this method were found to differ very little from those obtained using the surface angle method. There was a slight discrepancy in the range in parameter space for which solutions were obtained, but this was attributable to numerical error, and does not reflect any differences in the physical solutions being obtained. This is a significant result, since it provides circumstantial evidence that the breakdown of the solutions is not caused by the formation of waves on the free surface, and consequently some other mechanism must be found to explain this phenomenon.

As in the case of the first method and the work of Hocking and Forbes [10], and Tuck and Vanden Broeck [18], as the Froude number was increased small waves of numerical origin appeared on the free surface when breakdown was imminent.

6. Results and discussion

Using the results of the surface angle method, solutions were computed over a range of submergence depths and Froude numbers to map the region in parameter space in which such solutions exist. Figure 3 shows some typical solutions for the surface angle plotted against the mapped variable α over a range of Froude number and submergence depths of the source. In each case these are compared with the asymptotic solutions computed in section 3. For values of the Froude number smaller than those shown, the asymptotic solutions were almost graphically indistinguishable from the full nonlinear solutions. As in the asymptotic solution, the surface angle stays negative over the full interval for values of $\zeta_B > -1$, but becomes positive for some of the range for $\zeta_B > -1$. The dividing situation at $\zeta_B = -1$ corresponds to that in which the source submergence depth is almost exactly one half of the free stream depth.

The corresponding free surface profiles are shown in Fig. 4, revealing both solutions in which the free surface approaches the free stream level from above, when $h_S > \frac{1}{2}h_B$, and from below, when $h_S < \frac{1}{2}h_B$.

Figure 5 shows all of the solutions which are known to exist within the parameter space at the time of writing, with the exception of the supercritical stagnation point solutions of Mekias and Vanden Broeck [16]. In all cases, the pattern is very similar to that for the case of a line sink in a fluid of infinite depth. Stagnation point solutions exist for small values of the Froude number, F_B, up to some limiting value. There is no apparent reason for the breakdown of these solutions, but it is consistent over all computations attempted. There is then a range of Froude number within which there do not appear to be any steady solutions, before either a unique cusped solution, or a continuous range of cusped solutions is reached, depending on the geometry of the problem.

In the limit as $h_S/h_B \to 0$, i.e. $\zeta_B \to -\infty$ the limiting Froude number based on the source depth, F_S, should approach that obtained in Hocking and Forbes [10]. Figure 6 shows the limiting Froude number for each source depth, and despite the lessening of accuracy as ζ_B grows in magnitude, the solution appears to be approaching the correct value of $F_S \approx 1.4$.

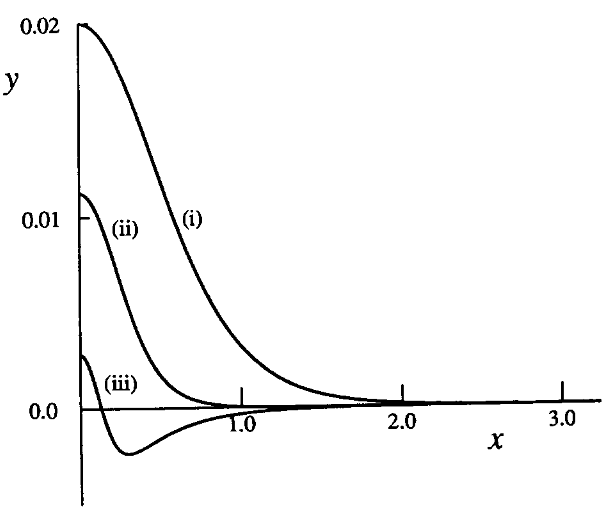

Fig. 4. Free surface profiles corresponding to the examples (i), (ii) and (iii) shown in Fig. 3.

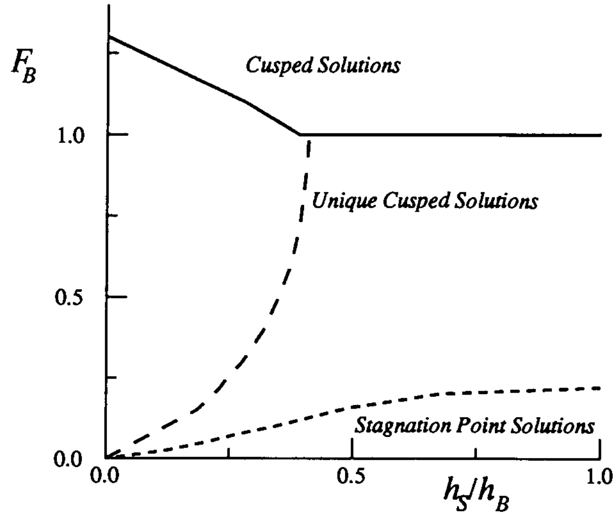

Fig. 5. Regions in the parameter space in which solutions have been computed. The new stagnation point solutions are those shown for $F_B \leqslant 1$.

In the region in which no solutions appear to exist, it is possible that there are steady solutions with waves on the free surface. The surface angle method used in this paper would not be able to compute such solutions because the mappings involved would introduce a singularity at $\alpha = 0$ if waves were present. Considerable effort was expended in an attempt to find solutions with waves using the second formulation, yet none were found, despite success in the past (see e.g. [3, 4]). In the end, these results merely serve to confirm those obtained using the surface angle technique. It is perhaps not surprising that the solutions should behave in this way, since the solutions in a fluid of infinite depth [10] broke down in a very

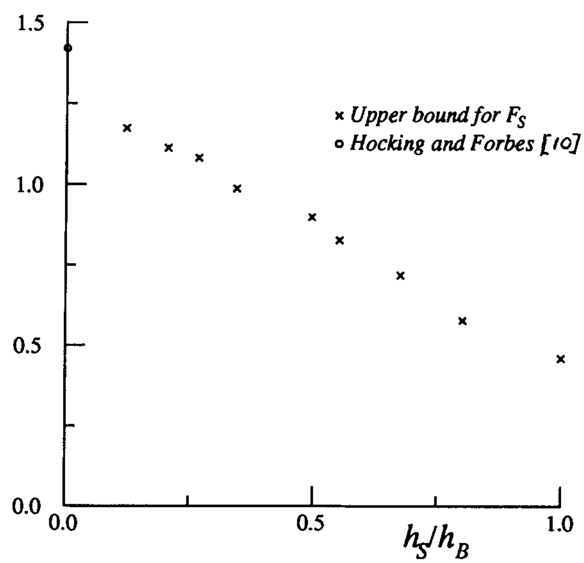

Fig. 6. A plot showing the numerically computed limiting values of the source depth Froude number, F_S, against the source to base depth ratio. The limiting value as $h_S/h_B \to 0$ shows reasonable agreement with that obtained by Hocking and Forbes [10].

similar manner, yet in that case there was no possibility of waves on the free surface a long way from the source because the fluid becomes stagnant in that limit.

The evidence therefore suggests that there are no solutions with waves, and once again we are left with no explanation for what is happening in this region, other than by analogy with the results of Forbes and Hocking [7] in which surface tension was included in the problem. These results are further complicated by the fact that for the case of the line sink on the bottom of the channel, experimental results suggest that the drawdown of the interface between two layers occurs at a Froude number of about 0.4 [11]. This places it somewhere between the maximum Froude number stagnation point solutions ($F_B \approx 0.23$) and the minimum Froude number cusped solutions at $F_B = 1.0$ [12, 19] which were once thought to characterise the critical drawdown.

One possibility is that the flows in this region are unsteady, and consequently a more sophisticated numerical approach would be required to obtain solutions. Work is continuing on these very difficult issues, but for the moment they must remain unresolved.

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